Math 246C Lecture 10 Notes

Daniel Raban

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1 Simply Connectedness of Universal Covering Spaces and Green's Functions

1.1 Simply connectedness of universal covering spaces

Last time, we were proving the existence of universal covering spaces.

Theorem 1.1. *Let X be a connected topological manifold. Then there exists a simply connected manifold $\tilde{X}$ and a covering map $p : \tilde{X} \to X$.*

Proof. Let $\tilde{X} = \{(x, [\sigma]) : \sigma \text{ is a path in } X \text{ from } x_0 \text{ to } x\}$. We have shown that $p : \tilde{X} \to X$ sending $(x, [\sigma]) \mapsto x$ is a covering map. We claim that $\tilde{X}$ is simply connected. When $|sigma$ is a path in X from x_0 to $x \in X$, consider the path in $\tilde{X}$: $\sigma' : [0, 1] \to X$ with $\sigma'(s) = (\sigma(s), [t \mapsto \sigma(ts)]) \in \tilde{X}$. Then $\sigma'(0) = (x_0, [\varepsilon_{x_0}])$ (w here ε_{x_0} is the constant path at x_0), and $\sigma'(1) = (x, [\sigma])$. Moreover, $p \circ \sigma' = \sigma$. So $\tilde{X}$ is path-connected.

Let σ'' be a closed path in $\tilde{X}$ with $\sigma''(0) = \sigma''(1) = (x_0, [\varepsilon_{x_0}])$. Then $\sigma := p \circ \sigma''$ is a closed path in X starting and ending at x_0. The path σ can be lifted to $\tilde{X}$, and by the uniqueness of lifts, σ'' sends $[0, 1] \ni s \mapsto (\sigma(s), [t \mapsto \sigma(st)]) \in \tilde{X}$. Thus, $(x_0, [\varepsilon_{x_0}]) = \sigma''(0) = \sigma''(1) = (x, [\sigma])$, so σ is null-homotopic in X. By the homotopy lifting theorem, σ'' is null-homotopic in $\tilde{X}$. $\square$

1.2 Green's functions in $\mathbb{C}$

We want to prove the uniformization theorem:

Theorem 1.2 (Poincaré, Koebe)**.** *Let X be a simply connected Riemann surface. Then X is complex diffeomorphic to $\hat{\mathbb{C}}$, $\mathbb{C}$, or the unit disc $D \subseteq \mathbb{C}$.*

Here is the starting point of the proof. We will try to construct a Green's function for X. Recall the notion of a Green's function for an open, bounded $\Omega \subseteq \mathbb{C}$ with C^2 boundary.

Definition 1.1. We say that $G(x, y)$ for $x \in \Omega$, $y \in \overline{\Omega}$ is a **Green's function** for Ω if

1. $G(x,y) = \frac{1}{2\pi}\log|x-y| + h_x(y)$, where $h_x \in C^2(\overline{\Omega})$ is harmonic in Ω.
2. $G(x,y) = 0$ for $y \in \partial\Omega$.

Remark 1.1. If G exists, it is unique. The function $y \mapsto G(x,y)$ is subharmonic in Ω. By the maximum principle, $G(x,y) < 0$ for all $(x,y) \in \Omega \times \Omega$.

Assume that $G(x,y)$ exists, and let $u \in C^2(\overline{\Omega})$ with $u|_{\partial\Omega}$. Cut out a small disc around x to get $\Omega_\varepsilon = \{y \in \Omega : |x-y| > \varepsilon\}$. By Green's formula,

$$\begin{aligned}\int_{\Omega_\varepsilon} (u(y)\Delta_y G(x,y) - G(x,y)\Delta u(y)) &= \int_{\partial\Omega_\varepsilon} \left(u(y)\frac{\partial G(x,y)}{\partial n_y} - G(x,y)\frac{\partial u}{\partial n_y}\right) ds(y) \\ &= \cancelto{0}{\int_{\partial\Omega}} + \int_{S_\varepsilon},\end{aligned}$$

where n is the unit outgoing vector, normal to $\partial\Omega_\varepsilon$, and $S_\varepsilon = \{y : |y-x| = \varepsilon\}$. Consider

$$\int_{S_\varepsilon} - \underbrace{G(x,y)}_{=O(\log(1/\varepsilon))} \frac{\partial u}{\partial n_y} \underbrace{ds(y)}_{=O(\varepsilon)} = O(\varepsilon \log(1/\varepsilon)) \xrightarrow{\varepsilon\to 0} 0.$$

Compute also

$$\begin{aligned}&\int_{S_\varepsilon} u(y)\nabla_y \left(\frac{1}{2\pi}\log|x-y| + h_x(y)\right)\frac{-(y-x)}{|y-x|}\,ds(y) \\ &\qquad = \int_{s_\varepsilon} u(y)\left(\frac{1}{2\pi}\frac{1}{|y-x|}\frac{y-x}{|y-x|}\frac{-(y-x)}{|y-x|} + O(1)\right) ds(y) \\ &\qquad = -\frac{1}{2\pi\varepsilon}\int_{s_\varepsilon} u(y)\,ds(y) + o(1) \\ &\qquad \xrightarrow{\varepsilon\to 0^+} -u(x).\end{aligned}$$

The left hand side in Green's formula equals

$$-\int_{\Omega_\varepsilon} G(x,y)\Delta u(y)\,dy \to \int_{\Omega} \xrightarrow{\varepsilon\to 0^+} \int_{\Omega} -G(x,y)\Delta u(y)\,dy,$$

where we can use the dominated convergence theorem since $G \in L^1_{\mathrm{loc}}(\Omega)$. We get

$$u(x) = \int_{\Omega} G(x,y) f(y)\,dy$$

if $f = \Delta u \in C(\overline{\Omega})$. Here, we have used that $u \in C^2(\overline{\Omega})$ and $u|_{\partial\Omega} = 0$.

Assume now that $u \in C_0^2(\mathbb{R}^2)$. Take $\Omega = D(0,R)$ for large $R > 0$, and let $x = 0$. Then

$$u(0) = \int G(0,y)\Delta u(y)\, dy = \int \left(\frac{1}{2\pi} \log |y| + h_0(y) \right) \Delta u(y)\, dy.$$

h_0 is harmonic in $D(0,R)$, so

$$\int h_0 \Delta u(y)\, dy = 0$$

after integrating by parts. So we get that

$$\int E(y)\Delta u(y)\, dy = u(0), \qquad E(y) = \frac{1}{2\pi} \log |y|$$

for all $u \in C_0^2(\mathbb{C})$. When this formula holds, we say that E is a **fundamental solution** of Δ, and we write $\Delta E = \delta_0$, where δ_0 is the Dirac measure at 0: $\delta_0(u) = u(0)$.

To construct $G(x,y)$ for a given Ω, we need to solve

$$\Delta_y h_x(y) = 0$$

in Ω with the boundary condition

$$\left(h_x + \frac{1}{2\pi} \log |x - \cdot| \right)_{\partial\Omega} = 0.$$

This can be solved using Perron's method. We will extend Perron's method to a Riemann surface and construct a Green's function using this method.